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Renormalization problem of the Yang–Mills theory with non-zero mass: The spectral approach

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Abstract. By using the spectral representation the vertex operator is investigated in lowest-order perturbation theory of massive Yang–Mills field. On examining the behaviour for large momenta the non-renormalizability of off-shell S-matrix for the massive Yang–Mills field is established.

Many attempts have been made on the problem of renormalizability of the theories of massive vector mesons, especially the Yang–Mills (Yang and Mills 1954) theory of a triplet of mutually interacting fields with non-zero mass. According to formal counting of singularities this theory is non-renormalizable; however, this power counting technique has been found to be unreliable for many cases (Boulware 1970). There are papers by Komar and Salam (1960), Kamefuchi and Umezawa (1961) and Boulware (1970) who have found this theory to be non-renormalizable. On the other hand, it was shown recently by Veltman (1968 and to be published) that, for certain diagrams with external particles on the mass shell, actual degrees of singularities are much less than those given by formal counting and the one-loop diagrams with more than four external vector boson lines are shown to be convergent

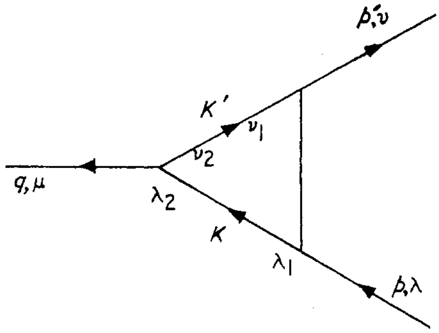

Figure 1. Vertex diagram.

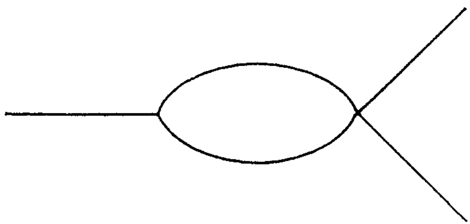

Figure 2. Vertex diagram.

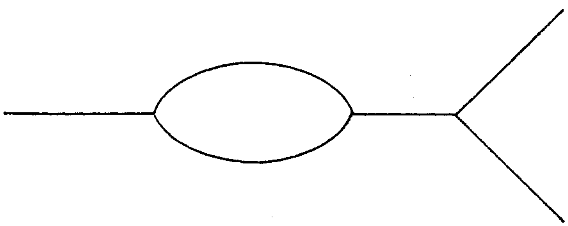

Figure 3. Vertex diagram.

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in the sense of renormalizability. Fradkin and Tyutin (1969) have also shown this theory to be renormalizable for diagrams with external particles on the mass shell. Most recently, a claim has been made by Slavnov (to be published) that this theory is renormalizable for any order of perturbation theory for the diagrams with external particles on the mass shell as well as *off the mass shell.*

To clarify this situation, a simple-minded calculation is made here for the lowest-order vertex operator using the dispersion theoretic technique and it is found that the vertex part shows sixth-order divergence. In order to proceed by the spectral method, the amplitude is decomposed into irreducible covariants multiplied by scalar functions of the momenta. Since the derivations from this point depend on the properties of these scalar functions only, we may proceed like scalar field theories without loss of generality. The reason the spectral technique is preferred is because the asymptotic behaviour, i.e. the dynamical properties of an amplitude, are best expressed through its analytic properties, and the value of the analytic function at any point is determined by the nearby singularities. For our vertex function according to unitarity the possible nearby singularity can be the normal threshold branch point corresponding to the two-particle intermediate state. The discontinuity across the branch cut can be represented by the figures 1, 2 and 3. For figure 1 the vertex function is given by

$$\frac{1}{2\pi i}\{M_{\lambda\mu\nu}(p, -q, -p')-\bar{M}_{\lambda\mu\nu}(p, -q, -p')\} = \frac{1}{\pi}\operatorname{Im} M_{\lambda\mu\nu}(p, -q, -p')$$

$$= \frac{1}{(2\pi)^3}\int dK\, dK'\delta(K-K'-q)\theta(K)\delta(K^2-m^2)\theta(K')\delta(K'^2-m^2)$$

$$\times\left(g_{\lambda_1\lambda_2}-\frac{K_{\lambda_1}K_{\lambda_2}}{m^2}\right)M_{\lambda_2\mu\nu_2}(K, -q, -K')\left(g_{\nu_1\nu_2}-\frac{K_{\nu_1}'K_{\nu_2}'}{m^2}\right)G_{\lambda_1\nu_1,\lambda\nu}$$

where $G_{\lambda_1\nu_1;\lambda\nu}$ denotes the amplitude for the scattering shown in figure 4 where the

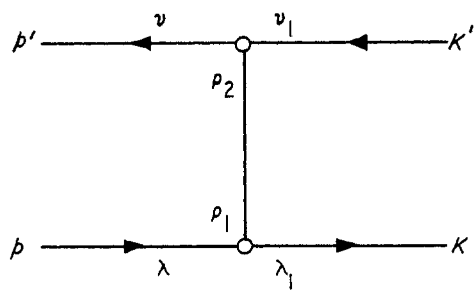

Figure 4. Scattering process $G_{\lambda_1\nu_2\lambda\nu}$.

function $M_{\lambda\mu\nu}$ can be expressed as

$$M_{\lambda\mu\nu} = \sum_i F_i(m^2, q^2)\chi^i_{\lambda\mu\nu}(p, q).$$

The F_i are the invariant amplitudes which are analytic functions of Lorentz scalar q^2. The $\chi^i_{\lambda\mu\nu}$ are the kinematic factors which carry all the spin indices. Because the spin of the particle does not take part in analytic behaviour, the dynamics is described by these scalar functions F_i only. We are considering two external particles on the mass shell, i.e. $p^2 = m^2 = p'^2$. This imposes two conditions on the vertex amplitude, the parity conservation and orthogonality of polarization vectors. These two considerations lead to only three independent amplitudes. Therefore we can write

$$M_{\lambda\mu\nu}(p, q) = F_1(q^2)p_\mu g_{\lambda\nu} + F_2(q^2)p_\mu q_\lambda q_\nu + F_3(q^2)(g_{\lambda\nu}q_\mu - q_\lambda g_{\mu\nu}).$$

On substituting for $M_{\lambda\mu\nu}$, the vertex function becomes

$$\frac{1}{\pi}\operatorname{Im} F_1(q^2)p_\mu g_{\lambda\nu} + \frac{1}{\pi}\operatorname{Im} F_2(q^2)p_\mu q_\lambda q_\nu + \frac{1}{\pi}\operatorname{Im} F_3(q^2)(g_{\lambda\nu}q_\mu - q_\lambda g_{\mu\nu})$$

$$= \frac{1}{(2\pi)^3}\int dK\, dK' \delta(K-K'-q)\theta(K)\delta(K^2-m^2)\theta(K')\delta(K'^2-m^2)$$
$$\times \Delta'_{\lambda_1\lambda_2}(K)\{g_{\mu\nu_2}(K'-q)_{\lambda_2} + g_{\nu_2\lambda_2}(-K'-K)_\mu + g_{\lambda_2\mu}(K+q)_{\nu_2}\}$$
$$\times \Delta'_{\nu_1\nu_2}(K')G_{\lambda_1\nu_1\lambda\nu}$$

$$\Delta_{\lambda_1\lambda_2}(K) = \langle T(U^i_{\lambda_1}U^j_{\lambda_2})\rangle = \frac{g_{\lambda_1\lambda_2} - K_{\lambda_1}K_{\lambda_2}/m^2}{K^2-m^2}\delta^{ij}$$

$$\Delta'_{\lambda_1\lambda_2}(K) = \left(g_{\lambda_1\lambda_2} - \frac{K_{\lambda_1}K_{\lambda_2}}{m^2}\right)\delta^{ij}$$

and

$$G_{\lambda_1\nu_1;\lambda\nu} = M_{\nu_1\rho_2\nu}\Delta_{\rho_1\rho_2}(p-K)M_{\lambda_1\rho_1\lambda}.$$

Now we will separate out the imaginary scalar function:

(i) Multiplying by $d_{\lambda\mu}(q)p_\nu$ we get

$$\left(\frac{4m^2-q^2}{4\pi}\right)\operatorname{Im} F_1(q^2) = \frac{1}{(2\pi)^2}\int dK\, dK' \ldots p_\nu d_{\lambda\mu}(q)$$
$$\times \{\Delta'_{\lambda_1\lambda_2}(K)M_{\lambda_2\mu\nu_2}\Delta'_{\nu_1\nu_2}(K')G_{\lambda_1\nu_1\lambda\nu}\}.$$

(ii) Multiplying by $q_\lambda d_{\mu\nu}(q)$ we get

$$\frac{3q^2}{\pi}\operatorname{Im} F_3(q^2) = \frac{1}{(2\pi)^3}\int dK\, dK' \ldots q_\lambda d_{\mu\nu}(q)(\Delta'_{\lambda_1\lambda_2}M_{\lambda_2\mu\nu_2}\Delta'_{\nu_1\nu_2}G_{\lambda_1\nu_1}).$$

(iii) Similarly, multiplying by the factor

$$\left\{q_\mu e_{\lambda\nu}(q) - \frac{q^2}{q^2-4m^2}p_\nu d_{\lambda\mu}(q)\right\}$$

we get

$$\frac{m^2q^2}{\pi}\operatorname{Im} F_2(q^2) = \frac{1}{(2\pi)^3}\int dK\, dK' \ldots \left\{q_\mu e_{\lambda\nu}(q) - \frac{q^2}{q^2-4m^2}p_\nu d_{\lambda\mu}(q)\right\}$$
$$\times (\Delta'_{\lambda_1\lambda_2}M_{\lambda_2\mu\nu_2}\Delta'_{\nu_1\nu_2}G_{\lambda_1\nu_1\lambda\nu})$$

where $d_{\lambda\mu}(q) = (g_{\lambda\mu} - q_\lambda q_\mu/q^2)$ and $e_{\lambda\mu}(q) = q_\lambda q_\mu/q^2$.

The scalar function $F_1(q^2)$ appears to be the most divergent one so we shall concentrate on the behaviour of this function. Performing K and K' integration we get

$$\frac{1}{\pi}\operatorname{Im} F_1(q^2) = \frac{\pi}{(2\pi)^3}\frac{-1}{\{q^2(q^2-4m^2)\}^{1/2}}\int_{-1}^{1} d(\cos\theta)Q$$

where

$$Q=\frac{-1}{t-m^2}\{-t^3(2q^4-8m^2q^2)-t^2(q^6+72m^6)+t(8m^2q^6-56m^4q^4-418m^6q^2)$$
$$+(2m^2q^8-24m^4q^6+44m^6q^4-200m^8q^2+608m^{10})\}$$

$$t=\frac{4m^2-q^2}{2}(1-\cos\theta).$$

Performing $\cos\theta$ integration, we get

$$\frac{1}{\pi}\operatorname{Im}F_1(q^2)=\frac{\pi}{(2\pi)^3}\frac{1}{24m^6}\left[\frac{1}{\{q^2(q^2-4m^2)\}^{1/2}}(q^8-96m^2q^6+564m^4q^4\right.$$
$$+2444m^6q^2+332m^8)+\frac{1}{8m^6}\frac{1}{\{q^2(q^2-4m^2)\}^{1/2}}(-4m^2q^8+30m^4q^6$$
$$\left.+28m^6q^4+1020m^8q^2-1350m^{10})\frac{1}{q^2-4m^2}\ln\frac{-m^2}{3m^2-q^2}\right].$$

Similar calculations are made for the other graphs. So for the imaginary part of scalar function $F_1(q^2)$ we get the following expressions:

$$\frac{\pi}{(2\pi)^3}\frac{1}{16m^4}\left[\frac{1}{\{q^2(q^2-4m^2)\}^{1/2}}(q^6+3m^2q^4-4m^4q^2-8m^6)\right]$$

and

$$\frac{\pi}{(2\pi)^3}\frac{1}{24m^4}\left[\frac{1}{\{q^2(q^2-4m^2)\}^{1/2}}\frac{1}{(q^2-m^2)}\right.$$
$$\left.\times(-7q^8-147m^2q^6-146m^4q^4-1256m^6q^2-592m^8)\right]$$

respectively, for the graphs (ii) and (iii). One can find the real values of the above function by writing down the dispersion relation considering the branch cut starting at $q^2=4m^2$.

Conclusions

The vertex function for the lowest-order Feynman diagram shows sixth-order divergence even on putting the two external particles on the mass shell. Thus we conclude that the theory of massive Yang–Mills fields is unrenormalizable for the diagrams with external particles *off the mass shell.* This result thus contradicts the claim made by Slavnov (to be published) of the renormalizability of the off-shell S-matrix. This result also differs from those of Komar and Salam (1960) who have found only a fourth-order divergence. The author is unable to discover the reason why the results differ.

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